Mathematics in its relation to other disciplines: Some examples related to economics and physics

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Resumen. Las matemáticas son a menudo vistas como una asignatura muy específica, ya sea por los estudiantes, los padres, los medios de comunicación o incluso los propios matemáticos. En muchos contextos, las matemáticas son temidas y vistas como una materia de selección, desconectado de aplicaciones interesantes de la vida real. Además, la estructura de las instituciones de enseñanza, en muchos casos, hace muy difícil la colaboración entre profesores de diferentes disciplinas. Al mismo tiempo, las matemáticas son cada vez más invisibles en la vida cotidiana, ya que la alta tecnología tiende a ocultar las matemáticas necesarias para su creación en sofisticadas cajas negras. En consecuencia, es todo un reto dar una respuesta adecuada a quienes, legítimamente, se preguntan para qué sirven las matemáticas. Nuestra propuesta en este trabajo es ver cómo las matemáticas en los diferentes currículos están realmente conectadas con otras disciplinas y hacer propuestas para hacer esta conexión más eficiente en beneficio tanto de las matemáticas como de otras materias. Los resultados que aquí se presentan provienen de diferentes trabajos de investigación que lideramos en relación con la física y la economía.

Palabras clave: matemáticas en otras disciplinas, física, economía.

Abstract. Mathematics is often seen as a very specific subject, either by students, parents, media or even mathematicians themselves. In many contexts, mathematics is feared and seen as a subject for selection, disconnected from interesting applications to real life. Moreover, the structure to teaching institutions, in many cases, makes the collaboration between teachers from different disciplines very difficult. At the same time, mathematics is more and more invisible in everyday life, since high technology tends to hide the mathematics necessary for its creation in sophisticated black boxes.

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As a result, it is quite a challenge to give an adequate answer to those who, legitimately, wonder what mathematics is useful for. Our propose in this paper is to see how mathematics in different curricula is actually connected to other disciplines and to give propositions to make this connection more efficient for the benefit of both mathematics and other subjects. The results presented here emerged from different research works that we lead in relation to physics and economics.

Keywords: mathematics in other disciplines, physics, economics.

1. Introduction.

Mathematics is often seen as a very specific subject, either by students, parents, media or even mathematicians themselves. In many contexts, mathematics is feared and seen as a subject for selection, disconnected from interesting applications to real life. Moreover, the structure of teaching institutions, in many cases, makes the collaboration between teachers from different disciplines very difficult. At the same time, mathematics is more and more invisible in everyday life, since high technology tends to hide the mathematics necessary for its creation in sophisticated black boxes. As a result, it is quite a challenge to give an adequate answer to those who, legitimately, wonder what mathematics is useful for.

Our purpose in this paper is to see how mathematics in different curricula is actually connected to other disciplines and to give propositions to make this connection more efficient for the benefit of both mathematics and other subjects. The results presented here emerged from different research works that we lead in relation to physics and economics.

After a short introduction presenting the situation in France, we will give two kinds of examples:

- In the context of French upper secondary scientific education: the use of vectors in physics and the connection between the notions of translatory and rotating movements in physics and the notions of translation and rotation in mathematics.
- In the context of French upper secondary and university education specialised in economics: the use of matrices for linear models of production and the use of functions, and especially of the concept of derivative and its relation to the notion of marginality in economics.

Our study reveals that, most often, teachers know very little about other subjects, even in relation to their own subject. Mathematics teachers do not want to get involved in too specialised applications while physics or economics teachers send their students back to their mathematics teacher for explanations on the use of mathematics in their field. As a result, students are used to seeing mathematics and other subjects as disconnected. This is reinforced by cultural differences, especially visible in the use of vocabulary or recipes that create artificial gaps between different disciplines.

2. The French context

Mathematics teaching in France has been marked by Bourbaki and the reform of modern mathematics. Even though a counter-reform took place about twenty years ago, French tradition tends to favour abstract and pure mathematics, rather than applications. However, due to different factors, among which social pressure and disinterest of students for mathematical studies have played an essential role, many mathematicians have now realised that mathematics teaching should give more space to applications and modelling in relation to other disciplines. Changes have then been implemented in curricula and syllabi, yet, with varied success.

One important feature of these changes concerns a global reform of the educational system with the implementation of new pedagogical devices proper to foster (impose) more inter-relations between different subjects. Devices have been introduced at different level of education. In lower secondary education, *Itinéraires De Découvertes* (IDD) is a specific time allotted to a class in order to work collectively on a multi-disciplinary subject, while in upper secondary education *Travaux Personnels Encadrés* (TPE) are projects led by small groups of students involving connections between at least two different disciplines. These projects are supervised by two teachers of different disciplines. In vocational education, *Parcours Pluridisciplianires à Caractère Professionel* (PPCP) concerns a project over a whole year involving a class or the whole school. Its goal is to study a professional problem according to different aspects in relation with different disciplines taught during the year. The necessity to implement these devices in all schools is part of the national curriculum. Teachers generally reacted favourably to these new propositions and many of them tried their best. However, many difficulties made things less successful than they could have been. It would be too long here to analyse these difficulties in detail, but they can be classified in two different categories:

- Structural difficulties due to a rigid centralised educational system, in which local initiatives are difficult
- Lack of training and cultural gaps between teachers of different disciplines.

Another feature of these changes concerns mathematical curricula. Previously mathematical curricula in upper secondary education were mostly designed from top to bottom. This means that after the curriculum for the scientific section was decided, the curricula for other sections were designed by dropping some parts of the scientific curriculum, in accordance with the time devoted to mathematics and the presumed lower ability of students in the different sections. In the late 80s, it was decided to design a curriculum specific for each section, in accordance with students' specialisation.

After this reform, the mathematical curriculum of the social and economic sciences (ES) section was designed in order to introduce more applications to economic and social sciences. In this sense, some parts of the curriculum were devoted to economic functions, marginality, elasticity, logarithmical derivative, percentage, statistics, etc. (see Gasquet, 1994 and Gasquet and. Chuzeville, 1994). More recently some notions about matrices and graph theory have been introduced. Therefore, according to official guidelines, the mathematical teaching in the ES section of upper secondary school should be more oriented towards applications in economic and social sciences and mathematical modelling. However, mathematics teachers were not prepared to this sort of change, and found it difficult to cooperate with their colleagues teaching economic and social sciences (due to a cultural gap). Our investigations, as well as several indicators, show that the success of this reform has therefore been limited by institutional and cultural constrains, this being reinforced by the absence of any specific training. For instance, in the recent pedagogical device of TPE (see above) very few projects involving mathematics and economic and social sciences emerged. In the ES section, a vast majority of projects involves economic and social sciences with history and geography, and when mathematics is involved, it is mostly for the use of statistical tools.

At the same time, in the mathematical curricula of all sections, the teaching of statistics has gained an increasing importance. The curriculum is not reduced to descriptive statistics, but also includes a component of inferential statistics through the notion of fluctuation of sampling. For each part of the syllabus, some topics are proposed for specific study, teachers being asked to select some of these, according to their students' particular interests. This change offered a possibility for less abstract and traditional mathematics.

More recently, general instructions have been given in all mathematical curricula in order to favour applications to different disciplines and discussions about modelling as often as possible. Again, these instructions have had differential effects and varied success. For instance in the curriculum of the scientific (S) strand, the exponential function was traditionally introduced as the inverse of the logarithm function. It is now demanded to introduce the exponential function at the beginning of the academic year, for improving the coherence between mathematics teaching and sciences teaching. The introduction of the exponential function is made, starting from the differential equation $f'=kf$, whose study "can be justified by one or two examples, for instance radioactivity treated in physics, or by the search for differentiable functions f such that $f(x+y)=f(x)f(y)$". In order to support this introduction, the accompanying document presents a text, which is the fruit of a common work of the groups of experts in mathematics, physics and biology.

As we have seen, mathematical teaching in France tends to be less abstract and to present more applications and modelling. We will now present examples of analyses and experiments we have made in two different types of research work.

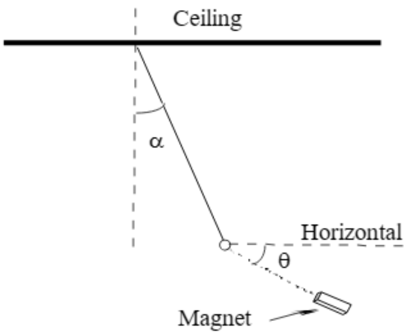
2.1 Examples of relations with physics

Vectors and forces

One interesting bridge between mathematics and physics in secondary education concerns the relation between vectors and forces. The notion of vector emerged in the middle of the nineteen century from different concerns involving purely mathematical problems as well as questions in physics (mostly electromagnetism) (Crowe 1967 and Dorier 2000). Vector is therefore by nature a concept in relation to both disciplines. How is this duality seen through teaching in mathematics and physics?

In mathematics, very few examples from physics are presented. Vectors remain essentially a tool for geometry and the teaching tends to focus on their algebraic properties. On another hand, vectors are a model of the concept of force in physics. However, various studies have shown that, although a force is characterised by a magnitude and a direction, tasks given in physics focus on the magnitude only (Genin et al. 1987 and Lounis 1989). Moreover a force is also attached to a point where the force is applied while a mathematical vector is invariant by translation. This duality is a source of difficulty for students.
We will not present here all the results of our analyses (Ba 2008, Ba et Dorier 2007 and to appear), but we will focus on a specific situation. This concerns a problem in physics (dynamics) designed in order to make the determination of the direction of a force essential for its solution.

Here is the text of the problem:



An iron small ball (comparable to a point M) with mass m is hung to the ceiling by a thread (whose mass will be neglected).

A magnet attracts the ball, the direction of the force makes an angle θ under the horizontal line (see drawing) and its magnitude is F.

When in equilibrium, the thread makes an angle α with the vertical (see drawing)

The only forces are: the weight of the ball, the attraction of the magnet and the tension of the thread.

Figure 1

Data: m=200g , θ=30° , F=2N, take g=10N/g.

1. Write the equilibrium equation.
2. Represent with the scale (1cm=1N) the forces in action.
3. What are the characteristics of the tension of the thread?

We now give the answers:

1. The equilibrium equation is given by the first fundamental law of dynamics:

$$\vec{F} + \vec{P} + \vec{T} = \vec{0}$$

2.

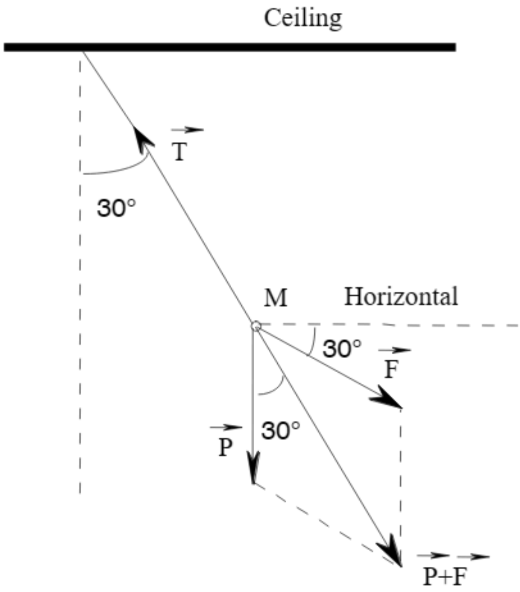


Figure 2

3. With use of relations in an isosceles triangle, it is easy to see that $\vec{T}$ makes an angle of 30° with the vertical and has a magnitude of $2\sqrt{2}$ N.

The interesting point in this problem, is that in question 2, one has to draw $\vec{F}$ and $\vec{P}$ first in order to draw $\vec{T}$ as the opposite of their sum. Then $\vec{T}$ gives the direction of the thread.

Therefore in order to draw the thread, one has to use the sum of two vectors, which is the essential key to the problem.

However, this task is problematic in the context of physics. Indeed, the construction of question 2 has to take place in a mathematical model, which is not reality. Moreover, in this model the point where the thread is attached to the ceiling can only be determined at the end of the process. Once this theoretical construction is made, one can come back to the drawing representing the reality and use the results of question 3 to represent the situation starting with the fact that the thread makes an angle of 30° with the vertical.

We have submitted this problem both to students and teachers in *Première S* (second scientific class of upper secondary school, age 15).

The students, tested in the physics class, did not have any problem with question 1. But they met real difficulties in question 2. They could not transfer the problem into the mathematical model. As a matter of fact, they did not see that there were two levels in the representation of the situation. On the other hand different studies show that students at this level have acquired sufficient knowledge about vectors to be able to draw the sum of two vectors and to answer questions like question 3, when given in a purely geometrical setting. This shows that students have sufficient competence in mathematics but are not able to mobilise it when necessary in physics. Moreover, they do not identify the mathematics at stake in a physics problem. The difficulty here is typical of modelling situations.

We asked physics teachers if they would give such a problem to their students, and if so what difficulty they think would appear. Massively, they admitted that this problem was close to a typical situation of dynamics, but at the same time they felt uncomfortable with the formulation. They did not believe that their students would handle the geometrical construction. For the solving of question 3, they also massively prefer a solution using projections on two orthogonal axes, which is a technique widely used in physics.

Mathematics teachers, on the other hand, would not be ready to give such a problem to their students because they do not consider it as part of mathematics. Moreover, the physics notions at stake are only taught one or two years after the sum of vectors is studied in mathematics.

This problem appears to be typical of the difficulty in building a bridge between mathematics and physics even when two notions are naturally related like vectors and forces. Teachers of both disciplines do not want to take charge of the link between the two and students cannot transfer their knowledge from one to the other. Only a joint effort from teachers of both disciplines can solve the problem. We are now working in this direction trying to build a teaching sequence involving the teachers of the two disciplines. However, we not only have to fight against reluctance to collaboration, but also to solve some difficult epistemological questions regarding modelling. It is also necessary to reduce the cultural gap between the two disciplines.

3. Translatory and rotating movement and translation and rotation

Another part of physics related to mathematical notions concerns translatory and rotating movements. The question is quite different from the previous case of vectors and forces. Indeed, here, the relation between physics and mathematics seems more obvious, since the same terms are used but, on the other hand, it is more mysterious, at least for what concerns translation. Indeed, it is well known that geometrical transformations are cognitively attached to dynamical representations. A mathematical transformation only takes into account an initial and a final state (i.e. an element and its image), but one often implicitly attaches an idea of movement between those two states. In this sense, the effect of a rotation on a geometrical object can be seen as a rotating movement of the object. This representation of a geometrical rotation is coherent with the concept of rotating movement in physics. However it is quite different with translation, since the dynamical representation of the translation of a geometrical object is attached to rectilinear translatory movement only and does not take into account all the other types of translatory movement studied in physics.

Indeed, in physics *an object is said to have a translatory movement when any segment attached to the solid remains parallel to itself during the movement* (def.1). Therefore, the trajectory of the object can be non-rectilinear, but follow any type of curve:

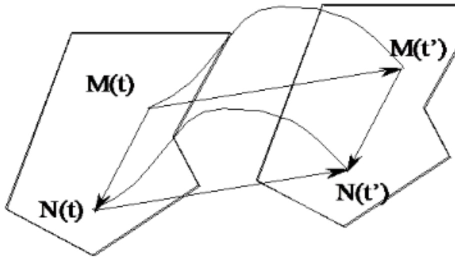


Figure 3

Experiments have been made involving mathematics and physics teachers about their representation of a translatory movement, and it shows that most mathematics teachers only think of rectilinear translatory movement and are totally puzzled when physics teachers try to explain what is a translatory movement by showing a movement with their hand following a non-rectilinear trajectory, yet with the hand remaining parallel to itself (Gasser 1996).

Another puzzling question is that most French physics textbooks (at the level of *Première S*), in the chapter introducing the definition of a translatory movement as given above, also give illustrations with objects on which vectors are drawn (like on the figure above), although the definition only mention segments. Indeed, the objects are always supposed not to change their shape during the movement, therefore a segment [M,N] on the solid cannot change its length. In a translatory movement any segment remains parallel to itself. So vector $\overrightarrow{MN}$ can have only two opposite directions, and cannot change length. Thus, according to a basic continuity principle, it is clear that vector $\overrightarrow{MN}$ cannot change its direction (because it would have to go from one direction to the opposite without being able to have any intermediary positions in between). Having the same direction and the same length it, therefore, remains identical.

In other words, a translatory movement can be characterised by the fact that *every vector on the solid remains identical (*def.2*).*

One can wonder why such a formulation is never used in physics, while vectors appear in practically all drawings. Certainly, the fear for being too abstract is the main reason.

This is the first proof of the distance separating physics and mathematics. Let us now see what the connection between translatory movements and mathematical translation can be and why this is neither explicit in physics nor in mathematics teaching.
Let us introduce the time in the notation, what physics teachers usually do not do at this level in order to avoid abstraction and formal notation. For each value t of $[0,T]$ (the duration of the movement) and any point M of the solid S, one calls $M(t)$ the position of the point M at time t. Then the definition of a translatory movement becomes:

A solid S has a translatory movement if, for any t, t' of $[0,T]$ and M, N of S, $\overrightarrow{M(t)N(t)} = \overrightarrow{M(t')N(t')}$ (def.3)

In terms of translation the condition can be expressed by:

A solid S has a translatory movement if, for any M, N of S, there is a translation τ_{MN} (independent of the time) such that for any t of $[0,T]$: $\tau_{MN}(M(t)) = N(t)$. (def.4)

Of course τ_{MN} is the translation of vector $\overrightarrow{MN}$. This is a first characterisation of a tanslatory movement using the mathematical notion of translation.

Moreover, if one applies what is sometimes known as the parallelogram rule (i.e. $\overrightarrow{M(t)N(t)} = \overrightarrow{M(t')N(t')}$ is equivalent to : $\overrightarrow{M(t)M(t')} = \overrightarrow{N(t)N(t')}$), one gets another characterisation of a translatory movement using the mathematical notion of translation:

S has a translatory movement if, for any t, t' of $[0,T]$ there exists a translation $\tau_{tt'}$ (independent of the point) such that for any M of S : $\tau_{tt'}(M(t)) = M(t')$. (def.5)

The difficulty here is that this translation does not give any information about the trajectory followed by the solid S between *t* and *t'*.

Finally, if, for distinct t and t', one divides the preceding equality by (*t'*-*t*), one gets:

$$\frac{\overrightarrow{\mathrm{M(t)M(t')}}}{\mathrm{t'-t}} = \frac{\overrightarrow{\mathrm{N(t)N(t')}}}{\mathrm{t'-t}}$$

Which becomes, when *t'* tends to *t*: $\overrightarrow{V_M(t)} = \overrightarrow{V_N(t)}$, which means that at any time during the movement all points have the same velocity.

Reciprocally, by integrating between *t* and *t*' the equality of velocity, one gets that: $\overrightarrow{M(t)M(t')} = \overrightarrow{N(t)N(t')}$).

This gives another characterisation of a translatory movement that students see in physics without any proof:

S has a translatory movement if, at any time, all points have the same velocity. *(*def.6*)*

In the teaching of physics in *Première S*, only definitions 1 and 6 are given to the students and no attempt to connect this to mathematical translations is made, either in books or by teachers (according to a questionnaire sent to a large number of teachers). Moreover, physics teachers either do no care about this connection or simply believe that translation and translatory movement are the same thing, while most mathematics teachers reduce translatory movement to the rectilinear case, in accordance with their dynamical representation of geometrical transformations.

Most students are used not to try to make bridges between physics and mathematics and therefore use the same word in two different disciplines without trying to find a connection. However, they have difficulties with translatory movements. They often get confused, for instance, between circular translatory movement and rotating movement. They also have difficulties in non-"classical" examples in making their definition operational when trying to prove that a given movement is translatory, while they have the mathematical skills at hand (Ba, 2003).

This situation is not satisfactory. Especially since students have all the necessary knowledge at hand to be able to understand with a minimum of time and work the different connections we have briefly established above. Again, the question is to know who, among the mathematics teacher and the physics teacher, should be in charge of making the connection explicit. Making this connection explicit would benefit physics teaching, of course, since it helps clarifying the notion of translatory movement, but also mathematics teaching, since it provides a use of vectors and translations in a rich context, with a challenging use of notations. For these reasons, we think that this should be a joint effort, either in parallel, in the mathematics class and the physics class, or even better, in a common session with both teachers. In a research work in progress (Ba, 2003) we are working on the design of such a teaching sequence and a training device in order to make a mathematics teacher and a physics teacher work together to try it out in their classes. The implementation will then be analysed with regard to students' work and teachers' direction of the situation.

3.1 Examples of relations with economics

The following examples are taken from our teaching (Dorier and Duc-Jacquet 1996) at the first year of university in Grenoble to students majoring in economics. The level of mathematical training in economics curricula in French universities is quite varied. Although it is of a very high standard in some prestigious universities, in the context of our experiment, it is much lower since our students are not specialised in quantitative methods. The first teaching sequence of the year is devoted to a situation designed to introduce some ideas about mathematical modelling (Dorier, 2006).

4. Matrices and linear models of production

Matrices are tools for linear models of production in economics. However, students usually find it difficult to deal with such formal objects and are easily overwhelmed by new definitions and rules of matrix algebra. The context of an example from economics may help them overcome some of these difficulties. In this section, we present an introduction to matrices using an elementary example from economics. This approach experienced several times with our students has proven to be efficient. It allows an introduction to matrices as objects for stocking data in a form that helps finding the pertinent numerical value but also facilitates computations in the model.

The example is based on an elementary linear model of production:

> A factory makes two kinds of threads T_1 and T_2 (the outputs) measured in reels using wool, acrylic, work and energy (the inputs) measured in, respectively, kg, kg, minutes and watts.
>
> The production is supposed to be linear, i.e. the quantity of inputs necessary is proportional to the quantities of output produced.
>
> - In order to produce one reel of thread T_1, one uses 1kg of wool, 0.6kg of acrylic, 1hour of work and 200w of energy.
> - In order to produce one reel of thread T_2, one uses 0.8kg of wool, 0.7kg of acrylic, 50mn of work and 170w of energy.
>
> What are the quantities of inputs necessary to produce 120 reels of T_1 and 80 reels of T_2?

Answering this question involves several computations. We give below the kind of reasoning necessary to find the quantity of wool necessary for the production :

- In order to produce 120 reels of T_1 one needs 120x1=120kg of wool.
- In order to produce 80 reels of T_2 one needs 80x0.8=64kg of wool.

So altogether, one needs 120+64=184kg of wool.

One needs to compute similar figures in order to get the quantity of the other inputs. After letting students manage these computations, blindly, one can ask them if they can identify some recurrent figures in the computations. This leads to the result that in order to find the quantity of each input, one needs to multiply the coefficients for T_1 by 120, the coefficient for T_2 by 80 and add the two results.

Once students have recognised this pattern, the teacher can explain that these computations can be formalised via matrices.

At first one introduces the matrix of production, which recapitulates in columns the quantities of inputs necessary for the production of one unit of each output:

$$A = \begin{array}{c} \begin{array}{cc} T_1 & T_2 \end{array} \\ \begin{pmatrix} 1 & 0.8 \\ 0.6 & 0.7 \\ 60 & 50 \\ 200 & 170 \end{pmatrix} \end{array} \begin{array}{l} \\ wool\ (kg) \\ acrylic\ (kg) \\ work\ (mn) \\ energy\ (w) \end{array}$$

It is important here to explain that putting the outputs in columns and the inputs in rows is purely conventional (arbitrary) but that everybody has to use the same convention.

Then the computations used in order to determine the quantity of inputs can be symbolised by the multiplication of A by the uni-column matrix of production:

$$\begin{pmatrix} 1 & 0.8 \\ 0.6 & 0.7 \\ 60 & 50 \\ 200 & 170 \end{pmatrix} x \begin{pmatrix} 120 \\ 80 \end{pmatrix} = \begin{pmatrix} 184 \\ 128 \\ 11\ 200 \\ 37\ 600 \end{pmatrix} \begin{array}{l} wool\ (kg) \\ acrylic\ (kg) \\ work\ (mn) \\ energy\ (w) \end{array}$$

This is a first step, that allows us to introduce the notation and an elementary example of multiplication, illustrating the 'multiplication of a row by a column law' in a context that gives meaning to students. In this approach, the matrix (of production) is not only a short presentation of data easy to grasp, but also a powerful computing tool.

Then, one can make the situation more complex:

The two kinds of thread are now used to make three kinds of material: M_1, M_2 and M_3 (the unit is a roll).
The matrix of production is:

$$B = \begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{pmatrix} 5.5 & 8 & 5 \\ 7 & 4 & 7.5 \end{pmatrix} \end{array} \begin{array}{l} \\ T_1 \\ T_2 \end{array}$$

Give the matrix C of production of the three types of material in relation to the wool, the acrylic, the work and the energy needed as inputs, without referring to the intermediary production of threads.

Here again, one needs to compute the data in a quite complex way, where it is easy to see recurrent patterns. Once the students have computed these data blindly, the teacher can ask to make the patterns explicit. He can then help students recognise the 'multiplication of a row by a column law' and state the rule for the multiplication of two matrices. Indeed, the matrix C can be modelled by the product AxB:

$$\begin{pmatrix} 1 & 0.8 \\ 0.6 & 0.7 \\ 60 & 50 \\ 200 & 170 \end{pmatrix} \begin{pmatrix} 5.5 & 8 & 5 \\ 7 & 4 & 7.5 \end{pmatrix} = \begin{array}{cc} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} & \\ \begin{pmatrix} 11.1 & 11.2 & 11 \\ 8.2 & 7.6 & 8.25 \\ 680 & 680 & 675 \\ 2290 & 2280 & 2275 \end{pmatrix} & \begin{array}{l} wool\ (kg) \\ acrylic\ (kg) \\ work\ (mn) \\ energy\ (w) \end{array} \end{array}$$

In this presentation, the multiplication of two matrices is therefore introduced in a context that gives meaning to conventional rules that many students have difficulty to accept in a purely mathematical context. It does not mean that all difficulties are avoided, but, at least, students have a richer background, suitable to motivate their learning and, furthermore, to make memorisation more efficient.

This presentation can also lead to an interpretation in terms of linear transformations. Indeed, the first level of production can be modelled via a linear transformation

$$f : q \in \mathbf{R}^2 \rightarrow q' = f(q) \in \mathbf{R}^4$$

that gives the quantities q' of inputs necessary for a production q of outputs. In other words, q represents the pairs of quantities of each thread to be produced and q', the quadruplet of the quantities of wool, acrylic, work and energy necessary for this production. f is a linear transformation whose matrix is A.

Similarly, there is a linear transformation g, modelling the second level of production:

$$g : p \in \mathbf{R}^3 \rightarrow q = g(p) \in \mathbf{R}^2$$

that gives the couple of quantities q of each thread necessary for a production of the triplet of quantities p of each material. g is a linear transformation whose matrix is B.

In order to get the linear transformation that gives the quantities q' of wool, acrylic, work and energy necessary for a production of the triplets of quantities p of each material, one needs to compose f and g:

$$fog : p \in \mathbf{R}^3 \rightarrow q = g(p) \in \mathbf{R}^2 \rightarrow q' = f(g(q)) \in \mathbf{R}^4$$

fog is a linear transformation whose matrix is the product $C=A\text{x}B$.

This example shows that, sometimes, purely mathematical concepts can be introduced in a context situated outside mathematics (here economics), in which the interpretation in terms of the other discipline gives a richer approach, suitable not only to giving more motivation to students, but also a consistent meaning that helps learning. When introduced only with reference to mathematics, matrices and the product of matrices may be seen as formal objects with arbitrary rules, while in the context of linear models of production, they appear as a suitable way to organise data referring to the model.

5. The concept of derivative in relation to the marginal function and its link with the average function

In economics, any economic function is related to two other functions:

- the *marginal function*
- the *average function*

The marginal function is defined by economists as the function measuring the change of the original function when its variable increases by one unit. For instance, the *marginal cost of production* measures, at each level of production, the increase of cost due to the production of one supplementary unit of output. Similarly, the *marginal productivity* measures the increase of production due to the use of one supplementary unit of input.

In other words, given any economic function f of the variable x (x being positive), the marginal function f_m is such that for any x:

$$f_m(x) = f(x+1) - f(x) \text{ or } f_m(x) = \Delta f(x), \text{ when } \Delta x = 1.$$

Like any economic function, the marginal function has a dimension. For instance in the case of a cost of production, expressing a monetary value as a function of a quantity produced, the marginal cost expresses a monetary value per unit of quantity (e.g. euros per kg). Therefore the dimension of the marginal function is (dimension of f) per (dimension of x). This shows that in the relation above there is a hidden division, i.e.: $f_m(x) = \frac{\Delta f(x)}{\Delta x}$, when $\Delta x = 1$.

Economists explain that the marginal function can be replaced by the derivative: $f_m(x) = f'(x)$. This substitution is very practical, since mathematics offers a theoretical framework for the derivative. However, it remains quite mysterious how one goes from the economic definition of the marginal function to the concept of derivative. Neither economics teachers, nor mathematics teachers, when they have to do it, know how to make the connection clear to their students. Our investigations, with teachers of both disciplines in secondary education and our analyses of textbooks show that this connection is widely used but remains problematic. Indeed, this question is typical of the gap that exists between the two subjects. In mathematics, functions are formal objects and are not related to measures of quantities. Moreover, variations can be small or big but in the concept of derivative they are infinitesimal, Δx tends to zero. In our problem, the key is a suitable choice of units. Indeed, for practical reasons, the units used for economic functions are such that quantities can be expressed by significant numbers, neither too small, nor too big. One would not choose a litre as a unit when dealing with problems involving production of petrol for instance, but rather a ton. Therefore, the units are usually chosen in such a way that a variation of one unit can be seen as a small variation. Under this condition, which is usually implicit, for $\Delta x = 1$, $\frac{\Delta f(x)}{\Delta x}$ is not very different from $f'(x)$ since Δx is small.

Obviously, in order to have a suitable explanation of the fact that the marginal function can be modelled by the derivative, one needs to take into account considerations from the economical reality (leading to a suitable choice of units) and from mathematics (for a small variation, the variation rate $\frac{\Delta f(x)}{\Delta x}$ is close to its limit when Δx tends to zero). If one does not refer to the two types of justification, one does not get a fully satisfactory explanation.

By definition, the average function measures the ratio of $f(x)$ over x:

$$f_M(x) = \frac{f(x)}{x}.$$

For instance, the average cost of production measures, at each level of production, the average cost of one unit of production or the average productivity measures the average quantity of output produced with one unit of input. Like the marginal function the average function has as its dimension: (dimension of f) per (dimension of x).

Therefore, these two functions can be compared, and indeed, there are some interesting economic results in their comparison. The most classical of these is that *when the marginal function is smaller than the average function, the average function decreases, and vice versa, when the marginal function is greater than the average function, the average function increases. Furthermore, when the two functions are identical, the average function has reached an optimum.*

This result can be illustrated by a very intuitive example. Let us consider a basket-ball team, and define f such that:

$$f(n) = \text{sum of heights of all } n \text{ players.}$$

If a new player joins the team, his height is the marginal height $f_m(n)$. If he is taller than the average height of the team, the average height of the team will increase, if he is smaller, it will decrease, and if he has exactly the average height, the average height will remain identical.
This example deals with discrete values of the variable, therefore it is not quite correct in terms of the model, yet, it gives an interesting intuitive illustration of the theorem, easy to memorise.
This theorem is taught in economics, using mathematical results on the derivative and illustrated with economic examples. The mathematical demonstration is simple; it lies on the following result, using the derivative of a quotient of two functions:

$$f_M{}'(x) = \left(\frac{f(x)}{x}\right)' = \frac{xf'(x) - f(x)}{x^2} = \frac{f'(x) - \dfrac{f(x)}{x}}{x} = \frac{f_m(x) - f_M(x)}{x}$$

Since x is positive, the sign of the derivative of the average function is the same as the sign of the difference: $f_m(x) - f_M(x)$. The theorem can be then easily deduced from this result.

However, an illustration, if not a proof, can also be given in a graphical context. Indeed the derivative measures the slope of the tangent to the curve, and the average function measures the slope of segment [OM], where M is the point of the curve with abscissa x: M = $(x, f(x))$.

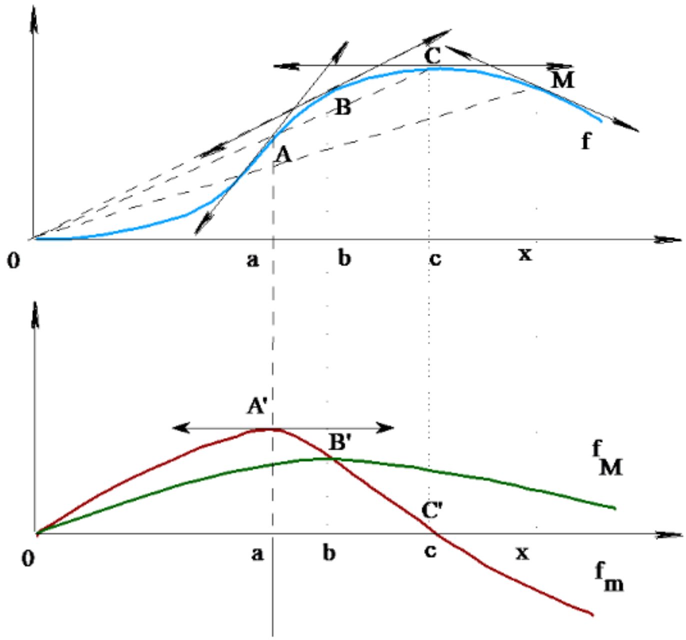


Figure 4

On this figure, we have represented a function f, and the corresponding marginal and average functions below. The graphs/curves representing f_m and f_M can be drawn by reading information from the graph of f.

There are three specific points: A,B and C, on the curve representing f, corresponding to the three points A', B' and C' on the curve of f_m and/or f_M.

- A is an inflexion point of the curve of *f*, i.e. the tangent crosses the curve, which changes its concavity in A. This corresponds to an optimum (here a maximum) of the derivative (or the marginal) function.
- B corresponds to the point, where the tangent to the curve representing *f* passes through O. Therefore, the marginal function equals the average function in b (abscissa of B'). The fact that the secant [OM] is tangent to the curve in B, also means that the slope of [OM] reaches its maximum in B. Therefore, the average function reaches its maximum in b, the abscissa of B' (or B). Before b, the slope of [OM] is smaller than the slope of the tangent, after b, it is bigger. Here, we have an intuitive graphical proof of the theorem of comparison between the marginal and the average functions.
- C corresponds to the maximum of the function *f*, it marks the change of sign of *f'*, the marginal function.

With students entering university, majoring in economics, mathematics teachers can use the context presented here in order to make a rich activity dealing with basic notions about functions and the concept of derivative, using graphical, algebraic and formal aspects, in relation to an economic interpretation. In France, the notion of derivative is taught in the last but one year of upper secondary school. However, very often, students entering university, especially if they do not come from the scientific section of upper secondary education, still have difficulty with this notion. The main idea here is to build bridges, not only between mathematics and economics but also between different settings at stake within mathematics. Several didactical studies have proven that cognitive flexibility is an important issue for the learning of mathematics. It is essential that students be able to interpret a result in graphical, algebraic or formal settings and to make connections between these settings.

In our example, after introducing the notions of marginal and average functions, in relation to the definitions seen in economics, with formal and algebraic mathematical interpretations, the teacher can start with the graphical representation, using a similar figure as above, asking the students to draw the shape of the curve representing the marginal and the average functions. The theorem on the comparison of the average and the marginal functions can be deduced from this specific example. Moreover, it can also be illustrated by the example of the basket-ball team. Then, the formal algebraic proof can be requested from the students.

We have experimented with such a didactical design several times with our students. It is striking how students who have a reputation for being reluctant to any formalism in mathematics are able to produce a correct formal proof of this theorem at the end of the instructional sequence.

Like in the example with matrices, the economic context offers a rich background in order to work with mathematical concepts. It helps giving more meaning and making interesting connections.

6. Conclusion

As we said in the introduction, the teaching of mathematics is subject to a social pressure that requires more applications and raises issues about modelling. The outside world forces mathematics to come out of its ivory tower. This is true for all levels of education in any context. However, it is even more essential for students whose major interest is outside mathematics. It is not possible anymore for mathematicians to remain isolated, away from applications, in a position of superiority. This is the best thing that could have happened to mathematics, which needs to become more visible. Our belief is that mathematics will not sell its soul by getting more interested in other disciplines. We hope to have shown with the few examples that we have sketched in this short paper, that by connecting itself to outside contexts, mathematics can be taught in a richer way, without reducing the value of its concepts. As we have shown in various ways in our four examples, the connection with other disciplines is also a way of making the formal aspect of mathematics accessible. Using a context issued from another discipline is not only a question of psychological motivation, but also an epistemological challenge. Indeed, using an example from another discipline, is not only a (fashionable) way to motivate students, but it is also a way to present a richer context where issues on the meaning of mathematics will automatically be addressed and questioned. This is not just an abdication of supremacy, but a humble recognition of the power of mathematics as a provider of models to other disciplines which has always been an essential part of its history.

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