cK¢, A MODEL TO UNDERSTAND LEARNER'S UNDERSTANDING

Discussing the case of functions

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Abstract: This text develops the invited talk I presented to the *international meeting on learning and teaching calculus to be held in Mexico in September 2015*. It addresses the problem of understanding and modelling students' conceptions taking as a theme the case of function. To set the *problématique*, the introduction reports the Arsac study of the development of the Cauchy's conception of uniform convergence. Then the issue of understanding students' understanding is discussed, and a framework is proposed: the model cK¢. Then conceptions of function across history and from a learning perspective are described with the tools provided by the model with a special emphasis on controls illustrating the key role they play.

Keywords: Calculus, Numerical Functions, cK¢, Conception, Misconception, Knowing, Milieu, Didactical Situations, Conceptual Fields, Learner Modeling, Design Experiment, Technology Enhanced Learning.

1. A short story as an introduction

Uniform convergence is a difficult concept which requires a good command of those of *function*, *limit* and *continuity*, and of the concept of *variable* as well. Arsac (2013) analyses this complexity when questioning the historical difficulty of reasoning on limits, starting from an analysis of the Cauchy's *Cours d'analyse*[1] published in 1821. It is in this textbook that the mathematician stated a first version of the theorem on the convergence of series of continuous functions (Cauchy, 1821, pp. 131-132):

> Let "(I) u_0, u_1, u_2 … u_n, u_{n+1}, &c…" be a series, then the theorem states:

> "Théorème[2]. Lorsque les différens termes de la série (I) sont des fonctions de la même variable x continues par rapport à cette variable dans le voisinage d'une valeur particulière pour laquelle la série est convergente, la somme s de la série est aussi, dans le voisinage de cette valeur particulière, fonction continue de x."

It is now known that this statement is not correct. The question is to understand why such an outstanding mathematician didn't realize the error he was making, and why it was so difficult to overcome it when

[1] http://gallica.bnf.fr/ark:/12148/btv1b8626657

[2] "When the various terms of series (1) are functions of the same variable x, continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum s of the series is also a continuous function of x in the neighborhood of this particular value." (trans. Bradley & Sandifer 2009 p.90)

counterexamples were provided? The Arsac's study of this episode of the history of mathematics is enlightening and full of lessons for mathematics educators.

A first thing Arsac invites the reader to notice is that the variable x is not explicit in the expression (I) of the series of functions, although the modern notation $f(x)$ was used in different parts of the *Cours d'analyse*. This may come from the fact that this representation was the usual representation of series of numbers, but it has also roots in the relations between *function* and *variable*, and the relation between *variables* and *quantities*:

> "When variable quantities are related to each other such that the value of one of the variables being given one can find the values of all the other variables, we normally consider these various quantities to be expressed by means of the one among them, which therefore takes the name the independent variable. The other quantities expressed by means of the independent variable are called functions of that variable." (trans. Bradley & Sandifer 2009 p.17)

In this quote, *variable* appears as an adjective and a noun, witnessing a tight relation between *variable* and *quantity*. This relation comes with a cinematic concept image of *limit* which origin, Arsac (ibid. p.17) reminds us, goes back to Neper and Newton. This concept image is reinforced by its relation to the graphical representation of functions as the one illustrated by Cauchy mathematical argument in support to the intermediate value theorem in the 1821 edition of his *Cours d'analyse* (but an analytic proof is proposed in a note[3]). The cinematic concept image is present in the definition of continuity in which a small increment of the variable produces a small increment of the function (dependent variable):

> "In other words, the function $f(x)$ is continuous with respect to x between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself." (trans. Bradley & Sandifer 2009 p.26)

As it is the case in the definition of *limit*, the evolution of the variable in the definition of *function* is conceived as a monotonous movement, and so is the conception of the evolution of the function (the dependent variable). As Arsac suggests it, this view is significant of the dominant understanding of the nature of *function* and *variable* at that time.

Then, in the expression of the series (I), u_n and x are two variables, x being the independent variable on which depends the functions u_n, but the former is left implicit de facto establishing – in the writing – a parallel between series of numbers and series of functions (i.e. the independent quantity and the dependent quantity).

The validity of the theorem on the convergence of series of continuous functions was backed by a narrative which expressed a qualitative reasoning of the same nature as that of the text of the definition of continuity.

[3] Cauchy, 1821, Note III, pp.460-520

"Denoting the sum of the convergent series

$$u_0, u_1, u_2, u_3, \ldots$$

by s and the sum of the first n terms [of the convergent series (I)] by s_n, we have

$$s = u_0+u_1+u_2+\ldots+u_{n-1}+u_n+u_{n+1}+\ldots$$
$$= s_n+u_n+u_{n+1}+\ldots,$$

and, as a consequence,

$$s-s_n = u_n+u_{n+1}+\ldots.$$

From this last equation, it follows that the quantities

$$u_n, u_{n+1}, u_{n+2}, \ldots$$

form a new convergent series, the sum of which is equal to $s-s_n$. If we represent this sum by r_n, we have

$$s = s_n+r_n,$$

and r_n is called the remainder of series (I) beginning from the *n*th term. Suppose the terms of series (I) involve some variable *x*. If the series is convergent and its various terms are continuous functions of *x* in a neighborhood of some particular value of this variable, then

$$s_n, r_n \text{ and } s$$

are also three functions of the variable *x*, the first of which is obviously continuous with respect to x in a neighborhood of the particular value in question. Given this, let us consider the increments in these three functions when we increase *x* by an infinitely small quantity *α*. For all possible values of *n*, the increment in s_n is an infinitely small quantity. The increment of r_n, as well as r_n itself, becomes infinitely small for very large values of *n*. Consequently, the increment in the function *s* must be infinitely small." (trans. Bradley & Sandifer 2009 89-90)

Arsac (ibid. p.58) notices that Cauchy did not introduce this text as a mathematical proof as the mathematician did for other theorems in his *Cours d'analyse*, but a "remark". The first lines fix the meaning of the symbols s, s_n and r_n as it would have been done for series of numbers, the fact that it is a series of functions is introduced after the notation by the sentence: "*suppose the terms of the series (I) involve some variable x*". As a matter of fact, what appears first, let say on the surface of the text, are numbers (i.e. variables representing quantities) and their dependence. This does not mean that it is what Cauchy meant, but here is a limit of his expression. There is also a vision of a monotonous movement of *x* and the effect it causes on the functions at each step of the reasoning. The increment of *x* is explicitly named – α – but this naming is not exploited as Cauchy could have done it. Things happen because they "*must*" happen.

Cauchy recognized that there are exceptions to the theorem as he formulated it in the 1821 publication of the *Cours d'analyse*, to which Abel and Seidel pointed the Fourier series counter-examples (Arsac ibid.chapter IV and V). He later modified the statement of the theorem and published it in a *Comptes rendus à l'Académie des Sciences* in 1853, introducing the condition:

$s_{n'}-s_n = u_n+u_{n+1}+\ldots+u_{n'-1}$ becomes infinitely small for infinitely large value of the numbers *n* and $n'>n$.

However, in this revised version of the theorem, the variable *x* remains implicit in the expression of the functions u_n. As Arsac (ibid. p. 61 sqq) points it, Cauchy refers to a number series which involves "some variable *x*" (formulation he chosen in the first formulation of the theorem). Having in mind the characterization of the convergence of series of numbers, he very likely did not envision expressing a definition of convergence specific to functions; instead, he manipulated numerical terms some of which being "variable quantities". His *démonstration* (mathematical proof), as he calls it now, is dominated by the use of natural language. This being associated to the implicitness of the variable x in the expression of the functions u_n, has important consequences: the role of the increment α is not addressed in the *démonstration*, the definition of "infinitely small"[4] favors a dynamic and monotonous concept image of convergence, the order of the appearance of the terms $\{n, x, \varepsilon\}$ driven by the rhetoric of argumentation is not congruent with the logical order. A consequence of the latter is that the dependence of n on ε and not on x, as it can be structurally evidenced by the modern algebraic expression[5], is – so to say – hidden.

The style of the Cauchy's revised version is still closer to a mathematical argument (a remark) than to a mathematical proof according to modern standards. There is no question that rigor is present as a willing[6], but it encounters obstacles: the definition of *variable* and *function*, the absence of *the sign* < and hence of computation on inequalities, the absence of a mathematical notation of *absolute value* (introduced by Weierstrass in 1841) and of the *quantifiers* (introduced at the turn of the XX° century); eventually natural language as a tool to express the reasoning on functions is infused by a cinematic concept image of *convergence* and the Leibnizian "*lex continuitatis*" (law of continuity[7]).

The Arsac's analysis of the Cauchy' understanding of function and convergence, is based on a critical and precise analysis of the original texts taking into account the situation of calculus in the first half of the XIX° century. It carefully avoids anachronism which could be introduced by rewriting the text with the language and formalization of contemporary mathematics. Such rewriting in modern terms hides the conceptual and technical difficulties mathematicians met to overcome them, and it leads to questionable interpretations as it was the case for Lakatos[8] which rewriting of Cauchy's mathematical texts suggests errors analogous to the ones students could make. But, more importantly it hides the difficulties coming from the conceptualization of the notion of *function* and *variable*.

This analysis of the difficulties encountered by mathematicians of the XIX° century faced with the counterexamples to the first formulation of the Cauchy's theorem of uniform convergence evidences the tight relation between *representation*, *language* and the *reasoning tools* on the one hand, and on the other hand the limits due to characteristics of the underpinning cinematic concept image of *continuity* and *limit*.

This short story illustrates the challenge of avoiding anachronism and over-interpretation as well as of taking into account contextual and situational characteristics of the analyzed mathematical content. What did Arsac for this historical case should also be done for mathematics of the classroom, mathematics of everyday life or ethnomathematics as well..

The key features of Arsac's approach can be synthetized along three lines of analysis. First, the characterization and description of the semiotic means available (language, symbols, diagrams), second, the elicitation of the reasoning rules as they are actualized by the discourse and the means for representation. To this should added, more hypothetically because they are generally left implicit in the discourse, the control structures which back the confidence and validity of judgements and choices made along the problem solving process.

[4] "*We say that a variable quantity becomes infinitely small when its numerical value decreases indefinitely in such a way as to converge towards the limit zero*". (trans. Bradley & Sandifer 2009 p.21).

[5] $\forall \varepsilon \ \exists N \ \forall n>N \ \forall n' \ [n'>n \rightarrow \forall x \ |s_n-s_{n'}|<\varepsilon]$

[6] But isn't it the case that rigor is always a willing?

[7] e.g. see (Crockett 1999)

[8] See Arsac ibid. p.62 sqq and 136-137.

Representation, operation, control are keywords of the model I designed in the mid-nineties for learner modeling in the framework of the theory of didactical situations (TDS, Brousseau 1986/1997[9]). It is this model which shapes the way I report here on the work of Arsac. In the following sections of this article, keeping with the case of calculus, I will present this model which original aim is to enhance our means to give account of mathematical understanding and competences.

2. Understanding understanding

The US common core state standard initiative[10] states well the problem research has to address: "*Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like?*" This question may have multiple answers depending on their frameworks and the background of the respondent. I will here present one which is built in the context of the TDS and the theory of conceptual fields (Vergnaud 1980/2009[11]).

These frameworks provide two postulates to ground an answer:

1. From a didactical perspective – "*Modeling a teaching situation consists of producing a game specific to the target knowledge among different subsystems: the educational system, the student system, the milieu, etc*". (Brousseau 1997 p.47)

2. From a developmental perspective – "*A concept is altogether: a set of situations, a set of operational invariants, and a set of linguistic and symbolic representations*." (Vergnaud 2009 p.94), what is referred to synthetically by the notation C=(S, I, $\mathscr{S}$)

Within the TDS theoretical framework, the teacher questioning the student's understanding is "*a player faced with a system, itself built up from a pair of systems: the student and, let us say for the moment, a 'milieu' that lacks any didactical intentions with regards to the student*" (Brousseau ibid. p.40). Whereas the TDS is explicit about models of didactical situations and has made progress on understanding their properties, it is less the case for the student<>milieu system. To make a progress in this direction, the Vergnaud theory of conceptual fields provides the first and fundamental elements for a possible solution. Its characterization has of concept has direct connections with the TDS description of the relation between a learner and a milieu based on different forms of knowledge (Brousseau ibid. p.61):

[1] The models for action governing decisions.

[2] The formulation of the descriptions and models.

[3] The forms of knowledge which allow the explicit "control" of the subject's interactions in relation to the validity of her statements.

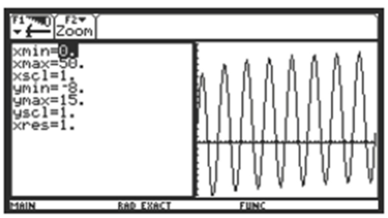
Apart from the set of situations S which is implicitly shared by both frameworks, the two other components, I and $\mathscr{S}$, of the Vergnaud's definition can be mapped onto the first forms of knowledge [1] and [2]. The difference between both approaches lies in the third form of knowledge [3] which brings to the fore knowledge as means of "control". This function of knowledge (resp. dimension of concept) was

[9] The first date indicates the original date of first publication of the ideas here referred to.
[10] [http://www.corestandards.org/math] retrieved 11/10/2013
[11] The first date indicates the original date of first publication of the ideas here referred to.

not absent from the Vergnaud model but not explicitly involved in his characterization. A mathematical theorem is both a tool and a statement: "if A then B" is a tool to obtain B if A is valid, it is also a statement which has a truth value. This duality of "*the operational form and the predicative form of knowledge*", as Vergnaud (2009 p.89 sqq) expresses it, facilitated keeping implicit the control dimension in the characterization he proposed. However, after Polya a long tradition of research on metacognition (e.g. Schoenfeld 1985 pp. 97-143) has shown the crucial role of control in problem-solving. Hence the suggestion to introduce explicitly "controls" aside the three components of the Vergnaud model.

Before proceeding to present an new version of a model of students understanding derived from the TSD and the theory of conceptual fields, it is necessary to clarify a vocabulary issue. I will use the term "conception" and not the term "knowledge" as it is classical in mathematics education.

Most research is based – more or less explicitly – on the hypothesis that learners act as rational subjects. But, one is often faced to rational thinking co-existing with knowledge which seems to lack coherency (from the observer's point of view). Let us take an example from the work of Trouche on the learning of calculus with graphic calculators:



Students are asked the following question:
« *f* is defined by f(x) = lnx + 10sinx. Is the limit +∞ in +∞ ? »
25% of errors were observed for students[12] using a graphic calculator, whereas without a graphic calculator there were no errors. (Trouche 1996 p.50)

Such a phenomena has been studied extensively, in particular contrasting mathematics practice in and out of school; what Lave (1988 p.63) recognizes as "*discontinuity of math performances between settings*". Bourdieu (1990) proposed a solution to this paradox: "*The calendar thus creates ex nihilo a whole host of relations […] between reference-points at different levels, which never being brought face to face in practice, are practically compatible even if they are logically contradictory*" (ibid. p. 83). The key elements are time on one hand, and on the other hand the diversity of situations. Time organizes the subjects' decisions sequentially in such a way that even contradictory, they are equally operational because appearing at different periods of their history: contradictory decisions can ignore each other. The diversity of the situations introduces an element of a different type. It is a possible explanation insofar as one recognizes that each decision is not of a general nature but that it is related to a specific sphere of practice (some may prefer to say that it is situated) within which its efficiency is acknowledged. *Within a sphere of practice students are coherent and successful; they are non-contradictory, but the sphere could be narrow*.

Contradictions (and failures) appear when students are faced with situations foreign to their sphere of practice but in which they have nevertheless to produce a response to a question, or a solution to a problem (e.g. as a requirement from the teacher). They mobilize what they have available which worked elsewhere, but more often than not this ends in systematically making errors. The classical position in the 80s was to consider these errors as symptoms of *misconceptions*. This term used to come with expressions like "*naive theory*", "*private concepts*", "*beliefs*" or even "*mathematics of the child*". Such views missed the fact that "*a child may not be 'seeing' the same set of events as a teacher, researcher or expert. […] many times a*

[12] 73 grade 12 students, scientific track (*Terminale S*)

child's response is labeled erroneous too quickly and […] if one were to imagine how the child was making sense of the situation, then one would find the errors to be reasoned and supportable" (Confrey 1990 p.29). Agreeing with this position, I renounced using the term "*misconception*". However, recognizing that learners may have different and possibly contradictory models-in-action to mobilize for (what we consider as) the same piece of knowledge. A word different from "knowledge" is needed because of the issue raised by the observation of possible contradictions in learners behaviors. One candidate is "conception" largely used in science education to refer to theory-in-action. More often than not the word conception functioned as a tool in discourses, not being taken as an object of study as such (Artigue 1991, p.266), although there was an acknowledged need (e.g. Vinner 1983, 1987) for a better grounded definition of conceptions, and for tools allowing analyzing their differences and resemblances.

In the two coming sections, I propose a definition of "conception", and then describe a model derived from the Vergnaud's triplet and pluggable into the TSD.

3. Behavior, conception and knowing

The only indicators one has to get an insight into learners' understanding are their behaviors and products which are consequences of the kind of understanding they may have engaged. Such evaluations are possible and their results are significant only in the case where one is able to establish a valid relationship between the observed behaviors and the invoked understanding. This relation has been relatively "hidden" as such for a long while as a result of the fight against behaviorism, but it has always been present in educational research at least at the methodological level. Indeed, the key issue is that *the meaning of a piece of knowledge cannot be reduced to behaviors, whereas meaning cannot be characterized, diagnosed or taught without linking it to behaviors*.

Being a tangible manifestation of the relationships between a person and her environment, a "behavior" depends on the characteristics of this person as well as on the characteristics of her environment. A now well documented example is that of an instrument which at the same time facilitates action if the user holds the required competence, and on the other hand limits this action because of its own constraints (Rabardel 1995, Resnick & Collins 1994, p.7). The words "person" and "environment", used here, refer to complex realities whose aspects are not all relevant for our investigations; this may be the case of the music preference of the person and the temperature in the room in which he or she stands, although we have always to be prepared to consider seriously features initially downplayed. What is of interest is the person from the point of view of his or her relationship to a piece of knowledge. For this reason I refer from now on to the learner as a reduction, if I dare saying so, of the person to her epistemic dimension. In the same way, I do not consider the environment in all its complexity, but only those features that are relevant with respect to a given piece of knowledge. Actually, this corresponds to the TDS concept of *milieu*, which is a kind of projection of the environment onto its epistemic dimension: *the milieu is the learner's antagonist system in the learning process* (Brousseau, 1997 p.57)

This situated perspective on *learner* and *milieu* suggests not considering understanding as a property which can be ascribed only to the learner but as a property of the interacting system formed by the learner and his or her antagonist milieu, to which I will refer as the *learner<>milieu system*. What is requested for this property to be valid is that the system satisfies the conditions required for it to be viable. It means that the system has the capacity to recover equilibrium after a perturbation which otherwise would cause its collapse, or that it can transform itself or reorganize itself. This is another formulation of Vergnaud's postulate that *problems* (perturbed system) are the sources and the criteria of knowing (Vergnaud 1981 p.220). It is important to realize that nothing is said about the process leading to the recovery of the equilibrium under the said constraints. They are proscriptive (Stewart, 1994 pp. 25-26), which means that they express necessary conditions to ensure the system viability, but not prescriptive, which means that they do not say in what way equilibrium must be recovered.

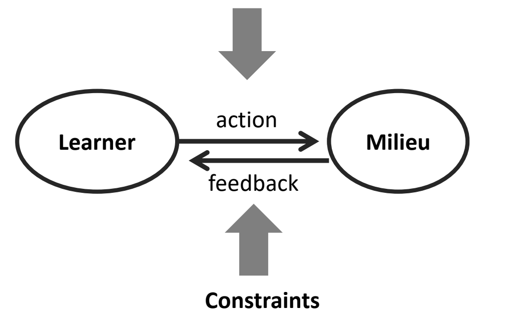


Figure 1. learner<>milieu system

Hence, a definition of *conception*:

A conception is the state of dynamical equilibrium of an action/feedback loop between a learner and a milieu under proscriptive constraints of viability.

The study and characterization of a conception will be based on observable behaviors of the system (action, feedback) and outcomes of its functioning. It requires evidence of the assessment of the equilibrium, which depends on the possibility to elicit the *learner's control* of the interaction and of the *milieu's reification* of failures and success by *adequate feedback*.

Geometry provides many good examples: constructing a diagram on a sheet of paper with a pencil is permissive to empirical adjustments, while dynamic geometry software allowing messing up a diagram by dragging points can reify the failure due to not conforming to geometrical properties (Healy et al. 1994) – although "students may modify the figure 'to make it look right' rather than debug the construction process" (Jones1999 p.254).

Indeed, this situated definition means that an observer may associate different conceptions to a learner<>milieu system[13] involved in a situation which characteristics he or she considers conceptually the same or involving problems he or she claims isomorphic. This is largely documented in the literature, for example by research on transfer, or by ethnomathematics research. Anyhow, in the observer's referential system, these different conceptions associated to the observed learner<>milieu system should be gathered in a common cluster. For this reason, *learner's knowing*[14] is defined as the set of conceptions which can be triggered by different situations the observer considers (mathematically) the same.

"Conception", "knowing" and "concept" – the latter being redefined later in the development of the model – are abstract terms which meaning is determined by their functions in the model and by the relations they have with other abstract terms in the related theoretical frameworks. Indeed, we must then discuss how far the proposed formalization makes sense when confronted with other use and context, or with (the perceived) "reality", and if they are adequate tools for the research they are meant to instrument.

I have associated to the model the name cK¢ which stands for "conception", "knowing", "concept". The following section outlines its main components.

[13] Often wrongly referred to as "student conception", for the sake of the simplification of discourse.

[14] I know that using "knowing" as a noun is uncommon, but it helps keeping distance with the word "knowledge" which has in education a strong authoritative connotation.

Key features of the model cK¢

The aim of a model is to provide a tool to establish links between theoretical frameworks which back it and the experimental field where will be set up experiments and carried out observations. It must be precise and effective tool to allow identifying what to observe, assessing the quality of data and performing an analysis.

The cK¢ model of students' understanding based on the TDS and the theory of conceptual fields, burrows the Vergnaud triplet but with a different vocabulary to avoid confusion with a psychological conceptualization and modelling. Actually, the aim of the model is not to provide a cognitive model as such, nor students' mental models as they are referred to in some research projects, but to characterize and represent *states* of the student<>milieu system. Two main differences with Vergnaud's model are: the use of the term "problem" instead of "situation", and the explicit introduction of "control structures". The meaning of "problem" is narrower than that of "situation", it refers to consequences of a perturbation of the student<>milieu system and not to the larger educational, institutional or material context in which it occurs.

Then the cK¢ formal characterization of a conception consists of a quadruplet (P, R, L, Σ) in which:

- P is a set of problems.
- R is a set of operators.
- L is a representation system.
- Σ is a control structure.

P proved to be more complex to elicit precisely than expected. Two opposite solutions have been proposed: (i) to include all problems for which the conception provides efficient tools (Vergnaud 1991 p.145), but for basic concepts this option is too general to be effective; (ii) to consider a finite set of problems from which other problems will derive (Brousseau 1997 p.30), but this option opens the question of establishing that such a generative set of problems exists for any conception. A solution familiar to most researchers consists of deriving P from both the observation of students in situations and from the analysis of historical and contemporary practices of mathematics. Actually, what one does when working on specific conceptions is to open a window on P by making explicit a few good representatives of its potential elements. These representatives work as kind of prototypical problems; this is a pragmatic implementation of Brousseau's proposal.

R corresponds to actions reified by behaviors one can observe during the functioning of the learner<>milieu system. They are not schemes in psychological terms but possibly data from which schemes may be inferred.

L refers to any semiotic tools which allow representing problems, supporting interaction and reifying operators. Actually, there is no difference there with the "*set of linguistic and symbolic representations*" Vergnaud includes in his definition.

Σ, the control structure, includes behaviors such as making choices, choosing operators, assessing feedback, making decisions, judging the evolution of a problem solving process. These metacognitive behaviors are more often than not silent and invisible, hence rarely accessible to observation. There are ways to overcome this difficulty by using specific experimental settings, for example inviting learners to work in pairs, with the expectation that this will be enough to elicit these behaviors as part of their verbal interactions; it is the objective of the TDS situations of formulations (Brousseau 1997 p.10 sqq)

It is worth noticing that the quadruplet is not more related to the learner than to the milieu with which he or she interacts: the representation system allows the formulation and the use of the operators by the active sender (the learner) as well as the reification of the actuators and feedback of the reactive receiver (the milieu); the control structure allows expressing the learner’s means to assess an action, as well as the criteria of the milieu for selecting a feedback. It is in this sense that the quadruplet characterizing a conception is congruent to the above conceptual definition of a conception as a property of the learner<>milieu system.

4. Outlines of the conceptions of “function” across its history

The word “function” may be associated to a number of different understandings. This is the case along the history of mathematics (Edwards, 1979; Kleiner, 1989; Kline, 1972; Smith, 1958), as well as along the mathematical life of learners (DeMarois and Tall 1996; Dubinsky and Harel 1992; Breidenbach et al. 1992; Thompson 1994; Sierpinska 1989; Vinner and Dreyfus 1989).

A first and efficient approach to distinguish these different conceptions of “function" in the course of the history of mathematics is to analyze them first from the point of view of the system of representation they implemented (Balacheff and Gaudin 2010).

One of the most ancient traces of the existence of function are tables and their uses. For example, Ptolemy (in the Almagest) knew that positions of planets change with time, and compiled astronomical numerical tables (Youschkevitch, 1976, pp. 40-42). Arabian astronomers in the 10th and 11th centuries also used precise tables. However, these tables did associate a given quantity to another one, and so, the idea of variable was not yet present.

The association of curves with tables leveraged the development of the concept of function, allowing making progress in formulating and solving the problem of determining the trajectories of the planets. Following Kline (1972), Kepler improved the computation of the position of planets essentially by adjusting geometrical curves and astronomical data, but without theoretical reference to explain why he considered the trajectories to be elliptical. The validity of the conjectured trajectories was then depending on the precision of the measurement of the planets’ positions and on the choice of a familiar geometric object: the ellipse. This permitted the description of the universe with simple mathematical laws. Kline also noted that most of the functions introduced in the 17th century were first studied as curves (ibid. p. 338), the geometrical trajectories of moving points (Kleiner 1989); hence the important role of geometry in this history.

The invention of the symbolism of algebra (Viete), and its development (Descartes, Newton, and Leibniz) was decisive: “*The evolution of the function concept can be seen as a tug of war between two elements, two mental images: the geometric (expressed in the form of a curve) and the algebraic (expressed as a formula)*” (Kleiner 1989). The separation of the study of functions from geometry is credited to Euler who published in 1748 an entirely algebraic treatise entitled “*Introductio in Analysin Infinitorum*”, without a single picture or drawing (Kleiner 1989, p. 284). “Function” was presented as the central object of Calculus. The analytic characterization of functions received a strong formulation by Euler, who asserted that a *function* is an *analytical expression* formed in any manner from a *variable quantity* and constants. In

1755, Euler formulated a general definition of function expressing the notion of dependence between variable quantities, and the notion of causality (Dhombres 1988, p. 45).

The function concept continued its development marked by the definition of Dirichlet which considered function as an arbitrary correspondence:

> "*y is a function of a variable x, defined on the interval* $a<x<b$ *if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established*" (quoted by Kleiner 1989 p.10).

This definition initiated a new "tug of war", this time between the algebraic conception and the logical one. The difficulties it brought along stimulated many discussions up to the 20th century (Monna 1972).

I will not go further into the history of the function concept but now limit my focus on three conceptions identified by the representation system on which they mainly rely: "Table", "Curve" and "Analytic"; let us refer to them as respectively C_T , C_C and C_A. Each of these three conceptions can be characterized by a quadruplet as it follows.

The *Table conception* C_T (P_T, R_T, Table, Σ_T) has essentially empirical grounds: the validity of a table depends on the precision of measurements and related computations under the requirements of a given experimental context. In the case of Kepler, for example, the validity must be evaluated against the quality of the interpolations and predictions that the ellipse allowed, as well as on the quality of the instruments available at that time. Therefore the corresponding control structure Σ_T was fundamentally of an empirical nature, providing the means that allowed the precision of tables to be verified with reference to the observations and to the measurements that had been carried out. However, the input/output table was the first means of representation used; it shaped quite a number of functions. Kline (1972, p. 338), reminds us that the table of the sine function was known with great precision long before the associated curve became a mathematical object. Then, the validity of the solution of a problem from the corresponding sphere of practice (P_T) did depend in an essential manner on the quality of rather concrete productions and actions necessary to collect and treat data.

- *Table conception* (P_T, R_T, Table, Σ_T),
  - P_T – Problems from physics and astronomy
  - R_T – Computation of ratio and integers, geometry
  - L_T/Table– Numerical tables, geometrical representation of curves, numbers, natural language
  - Σ_T – Confrontation between calculation and actual data

The *Curve conception* C_C (P_C, R_C, Curve, Σ_C) developed in the beginning of the 18th century, in response to the important problem of long distance navigation where coasts were out of sight. Thus, P_C originated in practical questions, and R_C included techniques of measurement, computation, and drawing. But the mathematical study of curves, as geometrical objects possibly associated to an algebraic expression, developed for itself including issues blending geometrical problems (e.g. like finding a tangent) to kinematic problems (e.g. velocities of points moving along a curve). Curves as geometrical entities were the ontological referent of this conception; if the word function was in use it was to refer to curves.

- *Curve conception* (P_C, R_C, Curve, Σ_C)
  - P_T – Study of curves as trajectories of points

R_T – Algebraic tools (since Euler) and manipulation of drawings
L_T/Curve– Representation of curves (not yet graphs), algebraic representation and natural language
Σ_T - Mathematical and experimental validation, mental experiments

The *Analytic conception* (P_A, R_A, Formula, Σ_A) follows a rupture in the epistemology of functions: function defined by an analytical expression does not need to refer to an experimental field (either natural phenomena or mechanical drawings). It can be studied for itself. This does not mean that modeling no longer plays any role; rather, it means that it is no longer central and does not characterize the conception. A purpose of the analysis of the 18th century (and of the 19th and 20th centuries) was the solution of functional equations, which were of great importance in physics (Dhombres, 1988), and the developments into infinite series which played a central role as operators (R_A) in those solutions. The corresponding control structure Σ_A depends on the specific characteristics of algebra as a representation system and on the operators it allows to implement. Computation of symbolic expressions and mathematical proof are the key tools to decide whether a statement is valid or not. Indeed, symbolic representations are not the only ones to be available and to be used. Following C_A, a function can be associated to a graph, that is, a set of pairs (x; y) in the Cartesian plane (where y is the value of the function for a given x). Graphical representations have a potential heuristic value by displaying phenomena that algebraic expressions do not easily evidence (for example, the intersection of two lines).

- *Analytic conception* (P_A, R_A, Formula, Σ_A),
  P_A – Study of functions (as objects)
  R_A – Algebraic tools
  L_A/Formula – Algebra, graphs
  Σ_A – Mathematical proof

This classification must not hide the complexity of the evolution of conceptions, their hybridization or cohabitation. On the contrary, the tension in the graphical register between graphs and curves was the origin of problems which stimulated the evolution. The general solution of partial differential equations expressing the vibrations of a finite string, subject to initial conditions, induced Euler to consider arbitrary functions that did not necessarily have analytic representations. New developments of the understanding of function took two centuries. The debate on what can count as function developed along the 19th century gave ground to the emergence of the Dirichlet *conception of function as a relation*.

5. Outlining the conceptions of “function”, the case of students

There is a large number of researches on secondary and post-secondary students understanding of function. I will refer here on a few seminal works, in particular research from Vinner, Dreyfus, Tall and Sierpinska, to illustrate the way the model contributes to clarify the different understandings.

The study of Vinner (1992) on students’ concept image of function is classical. Vinner (ibid. p.200) identified eight features of students’ ways of understanding function:

- “The correspondence which constitutes the function should be systematic, should be established by a rule and the rule itself should have its own regularities”;
- “A function must be an algebraic term”;
- “A function is identified with one of its graphical or symbolic representations”;

- "A function should be given by one rule";
- "Function can have different rules of correspondence for disjoint domains provided that these domains are regular domains (like half lines or intervals);
- "A rule of correspondence which is not an algebraic rule is a function only if the mathematical community officially announced it as a function";
- "The graph of a function should be regular and systematic";
- "A function is a one-to-one correspondence".

These features have been largely confirmed; they can be considered consensual nowadays as witnessed by the references to them in the contemporary literature.

Unlike history, students have some familiarity with Algebra when they are introduced to function. Moreover, most curricula provide them with some knowledge about the equation of a straight line and about the relation between graphical properties (intersection of lines) and algebraic properties (solution of an equation). Hence, the algebraic and graphical registers as well as their interactions play a central role; they make representation a privileged entrance point for the search for a characterization of the different conceptions. The nature of the relation between the algebraic and the graphical representation depends on what Sierpinska (1989) calls the synthetic views and analytic views:

> "*Curve analytical view*: a function is an 'abstract' curve in a system of coordinates; this means that it is conceived of points (x, y), where x and y are related to each other somehow." (ibid. 1989 pp.18).
>
> "*Curve, a synthetic view*: […] function is identified with its representation in the plane; it is a curve viewed in a concrete, synthetic way." (Sierpinska 1989 p.17).

Sierpinska added this clarifying comment: *"[the] relationship (between x and y, the analytical view) can be given by an equation. But the curve does not represent the relation. Rather, it is represented by the equation.*" (ibid.). Thus, the suggestion to consider two types of student conceptions: the *Curve-Algebra conception* (C_{CA}) and the *Algebra-Graph conception* (C_{AG}). Both conceptions share the same representation systems, algebraic and graphic, but with different interaction between both. In the case of C_{CA}, the criterion is that the curve must be represented by an equation; such a requirement is part of the corresponding control structure Σ_{CA}. In the case of C_{AG} the criterion is that the algebraic representation must be associated with a graph which one must be able to plot, a requirement which part of the respective control structure Σ_{AG}. The empirical distinction between C_{CA} and C_{AG} is not easy because their representation systems are very close the one to the other, when manipulating formulas and drawing diagrams. It is by looking at the control structures Σ_{CA} and Σ_{AG}, in relation to the operators and the way they are implemented, that the distinction can be shaped.

Ana Sfard (1991) would qualify these conceptions as operational conceptions of function because of their orientation towards a description of processes and actions. She emphasizes "*the deep ontological gap between operational and structural conceptions*" (ibid. p.4), characterizing the latter by the ability "*to recognize the idea 'at a glance' and to manipulate it as a whole, without going into details*" (ibid. p.4). As a matter of fact, recognizing this ability does not prevent the researcher from being able to characterize it empirically; which means the capacity to identify it by referring to empirical evidences. In order to address this issue and to give room to structural conceptions, Gaudin (2005 pp.97-98) introduces a *function-as-object conception* as the union of the operational conceptions which opens the possibility to trigger the most adapted operators, system of representation or control structure depending on the problem identified. The function-as-object conception includes controls managing the distinction between the representations and the so-called object as a whole, as it is defined by properties independent of specific processes and operations. In particular, these controls allow validating the correct resolution of a problem in other ways than the verification of the correctness of the processing of representations (ibid. p.98).

The next section illustrates the role of controls and their difference in nature taking the case of the *Curve-Algebra conception*, the *Algebra-Graph conception* and the *function-as-object conception.* It introduces the distinction between "*referent controls*" and "*instrumentation controls*" (Gaudin 2005 p.161).

6. The key role of controls

The identification of the controls enacted during a problem solving process is methodologically difficult. Whereas operators are accessible to an observer thanks to their reification by the behavior of the students, their interaction with the milieu and their actual productions, controls (e.g. reasons for a decision, criteria for a choice) are most often than not left implicit. It is by designing a situation of formulation, combining interactions with a milieu and social interactions, that there is a possibility to elicit them. Such situations, as defined by TDS (Brousseau 1997 p.10 sqq) as a situation of formulation, set constraints and instructions which make verbalization not only compulsory but necessary for the success of the task. An elementary situation of formulation consists of requiring from a group of students to solve collaboratively a problem and ensure an agreement on the solution.

Problems triggering a *function-as-object conception* (Sfard 1992) are more likely to give a key role to controls hence facilitating observing their role and their functioning. Among them problems of approximation are of a special interest because of the uncertainty on the criteria for the best approximation which requires an agreement among students on the features of the function and an analysis of the problem data. Smoothing problems, in particular, ask for consideration of multiple aspects in order to take decisions mobilizing qualitative as well as quantitative reasoning which resonate with the characteristics of the system of representation – either algebraic or graphical. The following case studied by Nathalie Gaudin (2005) has been designed on these principles. It exploits the functionalities of the Mapple software to provide a milieu within which students could ground an experimental strategy possibly triggering Curve-Algebra conceptions or Algebra-Graph conceptions. But the graphical systems of representation of these conceptions provide an insufficient qualitative support, and the algebraic representations lack the tools to assess the distance between functions and their regularity and shape, hence favoring the emergence of a function-as-object conception.

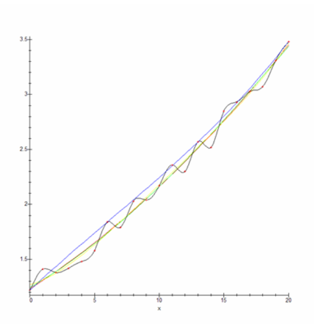
Here is the task Gaudin (2005 esp. ch.5) proposed to pairs of students[15] to achieve collaboratively using Mapple:

> *The following y_i provide values with possible errors (+/-10 %). These values come from a 3rd degree polynomial which coefficients are unknown, evaluated at a series of points x_i.*
> *Five approximations (f_1 … f_5) are proposed.*
> *You have to choose the one with approximate the best this polynomial:*
> - *on the interval [0;20]*
> - *on [0 ; +∞ [*

[15] 3 pairs of 2nd year student teachers, and 6 pairs of 2nd year students from an school of engineers.

Explain why you choose or not each of these approximations.

x_i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
y_i	1.22	1.41	1.38	1.42	1.48	1.58	1.84	1.79	2.03	2.04	2.17	2.36	2.30	2.57	2.52	2.85	2.93	3.03	3.07	3.31	3.48



$f1(x) = 1.2310 + 0.0752\,x + 1.789 \times 10\text{-}3\,x^2$

$f2(x) = 1.2429 + 0.06706\,x + 2.833\times 10\text{-}3\,x^2 - 3.48 \times 10\text{-}5\,x^3$

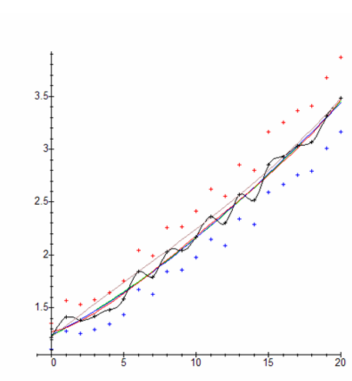
$f3(x) = 1.2712 + 0.0308\,x + 0.0115\,x^2 - 7.1626 \times 10\text{-}4\,x^3 + 1.704 \times 10\text{-}5\,x^4$

$f5(x) = 8{,}817\times 10\text{-}5x^3 - 0.00160x^2 + 0.10977x + 1.2200$
with $f5(0) = 1{,}22$; $f5(6) = 1{,}84$; $f5(13) = 2{,}57$ *et* $f5\ (20)=3{,}48$

f4 defined by: (1) it passes through each point *(xi, yi);* (2) on each interval *[xi ; yi],* it is a polynomial of a degree equal or less than *3;* (3) it is twice differentiable and its second derivative is continuous*;* (4) its algebraic representation is the following on each interval *[xi ; yi]): [3rd degree polynomials]*

The data gathered during the experiment, come from the observation of the students<>milieu interactions and from the verbal interactions between students. The first step in the analysis consists of identifying "atoms" (elementary aggregation of the raw data) allowing distinguishing between performed actions, statements about actions and statements about facts.

The methodological problem raised by the use of cK¢ – as it is the case of any research on verbal protocols – is of segmenting raw data to extract relevant items from the perspective of the analysis to be carried out and in line with the chosen framework. Here is an example taken from the case of Rémi and Olivier (Gaudin 2005 p.233 sqq):



Rémi: So the polynomial is somewhere there [A26]
Olivier: Yeah. The best approximation could be outside [A27 a]. So we have not made so much progress [A27 b].
Rémi: It depends how we define the best. It depends if you consider that a point out of there is a bad thing or if you consider it on average… if it is the set of point which ok… [A28] You see what I mean? So we try to draw all the polynomial, you see? We draw all
Olivier: all in a raw? [A29]
Rémi: Not sure that it will be easy to see anything, but we can try, and use the colors.
Olivier: You will remember that the yellow is the first? Can you write it? Then green… blue, we have to choose the colors… red. May be we avert yellow. Try « teal », it's the best color which exists [A30]

Atoms could be made of several utterances (e.g. A30) and one utterance may be split into several atoms (e.g. A28, A29). Once this treatment of the raw data has been achieved, atoms are classified depending on their roles.

Rémi: So the polynomial is somewhere there *Olivier*: Yeah. The best approximation could be outside.	A - assessment of a fact
Olivier: So we have not made so much progress.	B - judgment
Rémi: It depends how we define the best. It depends if you consider that a point out of there is a bad thing or if you consider it on average… if it is the set of point which, ok…	C - assessment of the judgement
Rémi: You see what I mean? So we try to draw all the polynomial, you see? We draw all *Olivier*: all in a raw?	D - decision on an action
Rémi: Not sure that it will be easy to see anything, but we can try, and use the colors. *Olivier*: You will remember that the yellow is the first? Can you write it? Then green… blue, we have to choose the colors… red. May be we avert yellow. Try « teal », it's the best color which exists.	E - assessment of an action

One observe that some controls are used to elicit the meaning of "approximation" or question it (e.g. C). They are important to stabilize the problem-solving strategy and the ground for decisions. They allow anticipating possible actions and checking their adequacy. They are the "*referent controls*" (Gaudin 2005 p.161). Once the referent controls have oriented the strategy, students must select the actions to perform; this is the role of the "*instrumentation controls*" (ibid.). Coupled with action they form an operator which structure is [*if* control *then* action].

This analysis confirmed the role of the curve-algebra and algebra-graph conceptions as starting states of the problem solving process, then the evolution towards a *function-as-object conception* which representation system includes algebraic and graphical registers in a fully integrated way, and which control structure includes controls on the function as such.

The following table summarizes these three conceptions which are not differentiated by the observed actions, but by the controls – referent or instrumental – which underpin them.

	Curve-algebra conception	**Algebra-graph conception**	**Function-as-object conception**
Referent controls	Global shape of the approximating curve Visual closeness of the approximating curve to the (x_i, y_i)	Closeness of the $f_j(x_i)$ and the y_i, or of the points (x_i, $f_j(x_i)$) and (x_i, y_i)	Global shape of the approximating curve and closeness of the $f_j(x_i)$ and the y_i or the points (x_i, $f_j(x_i)$) and (x_i, y_i)
Instrumentation controls	Related to the use of Mapple to plot the functions	Selecting the formula $[f_j(x_i) - y_i]^2$ Related to the use of Mapple for the calculations	Integration of the algebraic and graphical registers Full use of Mapple as a tool for Calculus
Representation systems	Mapple drawings and associated functionalities Algebraic formulas	Analytical and graphical	Analytical and graphical

Two types of controls drive the resolution of the problem, the referent controls and the instrumentation controls. The former implement properties expressed by the definition of approximation and allow anticipating the strategy and the criteria for an acceptable solution (Gaudin 2005 p.153). For example: [*if f is an approximation of P then f must follow the variation of P*], or [*the closeness of the $f_i(x_j)$ and y_j, and the position of the curve with respect to the (x_i, y_i)*]. The latter ensure the coherency between the referent controls and the actions to be performed; they drive the choices of operators.

A role of controls is to ensure that a conception is triggered within its domain of validity. Delineating this domain is necessary if one claims that students' conception are not contingent to circumstances and hence have all the characteristics of genuine knowledge. However, this is a challenging task; I address some aspects of it in the next section.

7. Conception and sphere of practice

Characterizing the sphere of practice of a conception is a difficult problem since the mathematical experience of learners is not restricted to the mathematical classroom. It is clear, that their spheres of practice are determined by activities outside the school as well as by what is done within the school in other disciplines than mathematics.

In the case of function, the curriculum even at the elementary level has a strong impact (Ayalon et al. 2017). Progressing in the course of their curriculum, students develop an understanding of functions which is more and more determined by the formal content taught and the actual everyday school practice. The choices made by textbooks to implement curricula are indicators of the learning contexts; their diversity is a first indicator of the possible diversity of students conceptions. Vilma Mesa (2009) has carried out an extensive comparative study[16] looking for conceptions possibly induced by mathematics textbooks which provides a picture of this diversity and its possible impact on learning. For this study, she has used cK¢ for the methodology it provides, not for the formalism – what is a fair use.

Vilma Mesa's analysis of her textbook corpus is guided by four questions parallelizing the four dimensions of the cK¢ characterization of conceptions. Given a task (exercise or problem):

- What use is given to function in the task?
- What does the student need in order to achieve the task?
- What representations are mobilized by the task?
- How could the student know that he or she has got a correct an answer?

The 2304 tasks coming from 35 textbooks (7th and 8th grade) were sorted following Biehler's taxonomy of "prototypical use of functions"[17] (Biehler 2005). An analysis grid were constructed and assessed based on a multiple judges approach, it is composed on the four lines of analysis of: 10 different types of problems (e.g. cause/effect relationship, graph defined relation, set-of-ordered-pairs relation), 39 operators (e.g. find percentage or number, find slope, name points on axis), 9 representation systems (e.g. arrow diagram, graph in two axes, symbolic, tabular), 9 controls (e.g. vertical line test, continuity assumed, use check points). Five types of conceptions are dominantly favored (Mesa 2009 p.86):

- *Symbolic rules* (20%): elementary tasks fulfilling a familiarization purpose, likely to induce what I above referred to as algebra-graph conceptions.
- *Ordered pairs* (14%): tasks requiring deciding whether or not a given ordered pairs, in the context of a mathematical or a non-mathematical situation, is a function or not. The representations are tables, sets of pairs, diagram or verbal.
- *Social data* (7%): task requiring appreciating a relation in a real-life context which provides meaning and content-based controls. It does not use algebraic representations of a function.
- *Physical phenomena* (4%): tasks based on the modeling of a time or a cause-and-effect relation. The controls are based on the content and context (mathematics or physics). It does not use algebraic representations of a function.
- *Controlling image* (3%): tasks in a context provided by a geometrical diagram, a graph, a numerical pattern or figural pattern. The few symbolic representations correspond to cases where symbols "acts as label" (like in the formula for the area of a rectangle).

The symbolic rules type of tasks is present in 71% of the textbooks, but they represent only 20% of the corpus meaning that a large practice of students are devoted to tasks in which function appears as a relation of dependence between two quantities. Tables, sets of pairs, diagram or verbal representations dominate, and apart the ordered pairs type of tasks, controls are context-based or even based on the didactical contract (e.g. Mesa 2009 p.65). The weight of the context in the controls favored by the tasks, instead of mathematical process oriented controls, opens the possibility for the development of different types of conceptions of function de facto fragmenting potentially students understanding.

[16] The corpus gathered 35 textbooks for seventh grade or higher in different languages (English, French, German, Portuguese and Spanish) which had specific sections devoted to functions.

[17] Natural laws, causal relation, constructed relations, descriptive relations and data reduction (Mesa 2009 p.11).

The balance between the different types of tasks discriminates fours clusters: rule oriented textbooks (50% of tasks are of a symbolic rule type), abstract oriented textbooks (78% of tasks are of a symbolic or ordered pairs type), abstract-oriented with application (includes at least one of the contextual type of tasks) and application-oriented (no symbolic rules or ordered type of tasks). It is remarkable that the abstract-oriented cluster contains only half or so of the textbooks, the other half includes the application-oriented cluster and a rule/abstract oriented cluster. Moreover, Vilma Mesa notices that "*the TIMSS items, as a set, do not share the same characteristics as those depicted by the tasks in the textbooks*" (ibid. p.99).

cK¢ worked as an efficient framework to evidence the diversity of the understandings of function potentially promoted by curricula. However, it has for such a study a heuristic role since textbooks are only indicators of what the implementation of curricula could be like. The actual life in the classroom framed by the teacher own understanding of function may give rise to a reality different from the one the analysis pictures. But this is a solid basis from which to go forwards.

The conceptions promoted by the different types of tasks are legitimate in the context of the practices as the textbooks suggest they could be. However Vilma Mesa reminds us (ibid. pp.114-115) that the conceptions she has identified have been reported by research on teachers' understandings of function, reinforcing their legitimacy. In other words, any of these conceptions are correct insofar as they allow achieving tasks and solving problems in the classroom context, and they satisfy the curriculum and teachers expectations and requirements. Even if some of them could turn into obstacles to overcome to progress in the understanding of function, they all contribute to its meaning. They can be considered as different facets of the concept, each situated from an epistemic and pragmatic perspective.

8. Conception, knowing and concept

The number of conceptions of function from an historical, didactical and epistemic perspective raises the question of their relations from a pragmatic perspective (e.g. efficiency, scope of use), and from a mathematical perspective (e.g. correctness, generality). Among these questions one is of a special importance: understanding learners' conceptions requires their interpretation from the perspective of the conception of the observer (e.g. teacher, researcher or evaluator). In particular, it is important to be aware of the fact that the generality or the falsity of a conception is not an intrinsic property but a type of relationship it holds with another conception. The case of Cauchy we reported in the beginning of this article illustrates how this relationship is susceptible to the translation from one system of representation to the other. This is often hidden by the fact that, as teachers or researchers, we tend to assess learners' productions and activities from the perspective of our own understanding.

Comparing conceptions assumes the more often than not hidden hypothesis that they are about the same mathematical content, what I will refer to as the *object of the conception*. The notion of mathematical object is difficult because of the immateriality of mathematical content; the ontological problem. This difficulty can be overcome within the model by, taking Vergnaud's postulate as a grounding principle: problems are sources and criteria of knowings (1981 p.220). Then, let C and C' be two conceptions and C_R the conception associated to the observer, so that it exists a representation mapping f: L→L_R and f': L'→ L_R. Then:

[*C and C' have the same object with respect to C_R if for all p from P there exists p' from P' such that $f(p)=f'(p')$, and reciprocally*]

Eventually, conceptions have the *same object* if their spheres of practice can be matched from the point of view of another conception which is in our case the conception of the researcher. The fact that two conceptions have the same object does not mean that they have another type of relationship (e.g. one being false with respect to the other, or more general, or partial, or else). It may be the case that some problems of P' (resp. P) cannot be expressed with L (resp. L'); and if they are, the translated problems may not be part of the sphere of practice of the other conception. The relation "To have the same object with respect to a given conception C_R" is an equivalence relation among conceptions with respect to C_R.

Let's now claim the existence of a conception Cμ more general than any other conception to which it can be compared. This seems to be a purely theoretical and abstract claim. Actually, it roughly corresponds to a conception of mathematics as it emerges from the practice of professional mathematicians. In our daily work as researchers or teachers, although in general left implicit, Cμ takes the form of a conception of reference we think shared by the research community mathematics educators. Then, the following propositions:

A "concept" is the set of all conceptions having the same object with respect to Cμ.

i.e. a *conception* is the actualization of a *concept* by a pair (subject/situation)

This definition is aligned with the idea that a mathematical concept is not reduced to the text of its formal definition, but is the product of its history and of all practices in different communities. Indeed, there is no agent holding the concept and no way to ensure that we can enumerate a complete list of these conceptions. So, a last definition will allow reducing the distance between this abstract definition and the needs we have to have a practical model:

A "knowing" is any subset of a concept which can be associated to a cognitive subject or a community.

i.e. a *conception* is the actualization of a *knowing* by a situation; it characterizes the subject/milieu system in a situation)

Given a concept, for example the concept of function discussed in this paper. Several different conceptions could form the knowing of this concept associated to an individual, each being enacted depending on various contextual features or problem characteristics. In the same way one may refer to the knowing of a mathematic classroom referring to the different conceptions likely to be enacted in this class. Eventually, one can refer to the XVIII° century knowing of function. These definitions of knowing and concept provide a framework which preserves students' epistemic integrity despite contradictions and variability across situations.

The name cK¢ comes from the names of the three pillars of the model: *conception*, *knowing*, *concept*. I keep the word "knowledge" to name a conception which is identified and formalized by an institution (which is a body of an educational system in our case).

9. Conclusion and additional comments

cK¢ proposes a framework for "learners modeling". Historically, it has been designed to take up the challenge of providing a model having an epistemic relevance to bridge research in mathematics education and research on educational technology. On the one hand it had the objective to offer a common framework to express the knowledge base on learners understanding of mathematical concepts; on the other hand it intended to respond to the need for representations both understandable by researchers in mathematics education and computationally tractable. The formalism it dares should enhance the way one informs the design of technology enhanced learning environments, complementing descriptions generally available in natural language with no standardized narrative structure.

Research in mathematics education develops jointly theories and experimentations, in this context models can serve as mediators between theories of which they require an articulate and precise understanding, and experiments of which they frame the design and drive the collection of data. However, both theories and experiments raise difficult issues. On the side of theories, one has to deal with a complex discourse which rarely makes explicit all details and hence gives room to non-univocal interpretations. On the side of experiments, the practical implementation is always richer and more complex than what the design of models anticipates. Moreover, in the case of conceptions, one is confronted with issues (that Toulmin already noticed when proposing a model of argumentation): distinguishing operators from controls is not absolute (e.g. theorems can be activated as tools or predicates), and controls are more often than not implicit. Such difficulties require further theoretical as well as methodological investigations.

The case of function evidences the complexity of making sense of students' understanding. against what history teaches us about the evolution of this concept. And indeed we would be very cautious with the idea that the "historical study of the notion of function together with its epistemological analysis helped us to analyze the student' mathematical behavior" (Sierpinska 1989 p.2). It is clear that the epistemological analysis is an essential tool, but the historical analysis may induce a view of the notion of function which hides the role played by the modern school context. The historical analysis could delineate the notion from the mathematical point of view, from the epistemic point of view we must be prepared to see things in a rather different way. Actually Sierpinska acknowledged that "the students' conceptions are not faithful images of the corresponding historical conception" (ibid. p.19). For example, one of the questions one has to consider is that of knowing what could be the essential difference between students' algebraic conceptions and the "corresponding" historical conceptions. It is also striking that tables play a very limited role if any at all in the situations involving functions: if they are present it is in relation to concrete situations in which the aim is less to analyze a function than to analyze data (the function is seen as a tool for data analysis).

Initially based on the Theory of Didactical Situation and the Theory of Conceptual Field, the cK¢ modeling framework is not restricted to them. For the purpose of its development and in order to enhance its efficiency it is necessary to integrate other theories to strengthen its components (e.g. representation, control system). But cK¢ holds other promises. It facilitates building a bridge between knowing and proving, constructing a link between control and proof, hence facilitating understanding the relation between argumentation and proof.

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